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LETTER TO THE EDITOR

Landau levels on the hyperbolic plane

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Abstract
The quantum states of a spinless charged particle on a hyperbolic plane in the presence of a uniform magnetic field with a generalized quantization condition are proved to be the bases of the irreducible Hilbert representation spaces of the Lie algebra $u(1,1)$. The dynamical symmetry group $U(1,1)$ with the explicit form of the Lie algebra generators is extracted. It is also shown that the energy has an infinite-fold degeneracy in each of the representation spaces which are allocated to the different values of the magnetic field strength. Based on the simultaneous shift of two parameters, it is also noted that the quantum states realize the representations of Lie algebra $u(2)$ by shifting the magnetic field strength.

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1. Introduction

The Landau levels problem appears to be related to rich physical and also mathematical aspects, which is worthwhile studying in various possible configurations. The Landau levels problem on a hyperbolic plane has been studied in various papers [1–4], and also been extended to the noncommutative case [5, 6]. In this letter, in order to consider the Landau levels problem on a hyperbolic plane we use simultaneous laddering relations with respect to two different parameters of the associated Gegenbauer functions. It must be emphasized that the laddering relations with respect to one parameter are equivalent to the sense of shape invariance from the point of view of Gendenshtein [7]. Simultaneous laddering relations with respect to two different parameters whose formulation was first performed for the associated hypergeometric and Jacobi functions [8, 9], and then for the associated Laguerre functions [10], provide rich algebraic structures for the special functions and their corresponding differential equations. See [11–16] for some previous works on these fields, but with different emphasis. As an example in [11], using two different types of the laddering relations realized simultaneously by the associated Gegenbauer functions, we studied some aspects of quantum splitting corresponding to the motion of a free particle on a hyperbolic plane.

In this letter we extract the solutions of Schrödinger equation corresponding to the motion of a spinless charged particle on the hyperbolic plane in the presence of a uniform magnetic field (i.e., Landau problem) for which a generalized quantization condition of the magnetic field is established. It is shown that for each of these quantized values of the magnetic field using the solutions we can construct the Hilbert representation spaces of the Lie algebra $u(1,1)$ which describe the dynamical symmetry group $U(1,1)$ with infinite-fold degeneracy. In fact, the representation of the Lie algebra $u(1,1)$ is realized when we deal with the Landau levels corresponding to a given constant value of magnetic field strength. Also, using the simultaneous laddering relations with respect to two different parameters we obtain the representation of the Lie algebra $u(2)$ by the quantum states. It is shown that the representation is realized when the Landau levels corresponding to the different strengths of the uniform magnetic fields are used.

2. Two different laddering relations for the associated Gegenbauer functions

As has been shown in [9], for a given real parameter $\lambda > -1$ and all integers $n \geqslant 0$ and $0 \leqslant m \leqslant n$, the associated Gegenbauer functions $P_{n,m}^{(\lambda)}(x)$ with the Rodrigues representation

$$P_{n,m}^{(\lambda)}(x) = \frac{(-1)^m}{2^n\Gamma(\lambda+n+1)}\sqrt{\frac{\Gamma(2\lambda+n+m+1)}{\Gamma(n-m+1)}}\frac{1}{(1-x^2)^{\lambda+\frac{m}{2}}}\frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(1-x^2)^{\lambda+n} \tag{1}$$

satisfy an orthogonality relation with respect to the weight function $(1-x^2)^\lambda$ in the interval $-1 < x < +1$ as

$$\int_{-1}^{1} P_{n,m}^{(\lambda)}(x)P_{n',m}^{(\lambda)}(x)(1-x^2)^\lambda\,\mathrm{d}x = \delta_{nn'}h_n^2(\lambda), \tag{2}$$

in which $h_n(\lambda)$ as the norm of the associated Gegenbauer functions $P_{n,m}^{(\lambda)}(x)$ is independent of the parameter m:

$$h_n(\lambda) = \frac{2^{\lambda+\frac{1}{2}}}{\sqrt{2\lambda+2n+1}}. \tag{3}$$

It is clear that the term $1+(-1)^{n+n'}$ becomes zero when one of the integers n and n' is even and the other one is odd. In this case, $\delta_{nn'}$ will be zero as well. In other cases, the term $1+(-1)^{n+n'}$ takes the value 2 which, in turn, leads to the values 0 or 1 for $\delta_{nn'}$. Thus, considering the relation $P_{n,m}^{(\lambda)}(-x) = (-1)^{n-m}P_{n,m}^{(\lambda)}(x)$ and using relation (2) we can conclude the following orthogonality relation for the interval $0 < x < +1$ with the same weight function $(1-x^2)^\lambda$:

$$\int_{0}^{1} P_{n,m}^{(\lambda)}(x)P_{n',m}^{(\lambda)}(x)(1-x^2)^\lambda\,\mathrm{d}x = \delta_{nn'}\frac{h_n^2(\lambda)}{2}. \tag{4}$$

The laddering relations with respect to n for a given m, and also with respect to m for a given n, are represented by the associated Gegenbauer functions $P_{n,m}^{(\lambda)}(x)$ as [9]

$$A_+(n;x)P_{n-1,m}^{(\lambda)}(x) = \sqrt{(n-m)(2\lambda+n+m)}P_{n,m}^{(\lambda)}(x) \tag{5a}$$

$$A_-(n;x)P_{n,m}^{(\lambda)}(x) = \sqrt{(n-m)(2\lambda+n+m)}P_{n-1,m}^{(\lambda)}(x) \tag{5b}$$

and

$$A_+(m;x)P^{(\lambda)}_{n,m-1}(x) = \sqrt{(n-m+1)(2\lambda+n+m)}\,P^{(\lambda)}_{n,m}(x) \tag{6a}$$

$$A_-(m;x)P^{(\lambda)}_{n,m}(x) = \sqrt{(n-m+1)(2\lambda+n+m)}\,P^{(\lambda)}_{n,m-1}(x), \tag{6b}$$

respectively, in which the explicit forms of differential operators $A_+(n;x)$ and $A_-(n;x)$ as well as $A_+(m;x)$ and $A_-(m;x)$ are given by

$$A_+(n;x) = (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} - (2\lambda+n)x \tag{7a}$$

$$A_-(n;x) = -(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} - nx \tag{7b}$$

and

$$A_+(m;x) = \sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(m-1)x}{\sqrt{1-x^2}} \tag{8a}$$

$$A_-(m;x) = -\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(2\lambda+m)x}{\sqrt{1-x^2}}. \tag{8b}$$

In fact the choice of (1) for the normalization coefficients in the Rodrigues formula has allowed us to separate the associated differential equation as two different types of laddering relations with respect to the indices n and m. Now we can obtain one pair of the raising and lowering relations in order to provide the tools necessary for describing the Landau levels on hyperbolic plane with the dynamical symmetry group $U(1,1)$. This will be another one of the results of the idea of simultaneous shape invariance with respect to two different parameters n and m. For this purpose, firstly, we define one pair of laddering operators as

$$\begin{aligned} A_{\pm,\pm}(n,m;x) &:= \pm[A_\pm(m;x), A_\pm(n;x)] \\ &= \mp x\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(\lambda\pm\lambda+n)x^2 - 2\lambda - n - m + \frac{1}{2} \pm \frac{1}{2}}{\sqrt{1-x^2}}, \end{aligned} \tag{9}$$

where the explicit forms of them are calculated by using equations (7a), (7b), (8a) and (8b). Applying equations (5a), (5b), (6a) and (6b), one may derive the simultaneous laddering relations with respect to two parameters n and m as follows:

$$A_{+,+}(n,m;x)P^{(\lambda)}_{n-1,m-1}(x) = \sqrt{(2\lambda+n+m-1)(2\lambda+n+m)}\,P^{(\lambda)}_{n,m}(x) \tag{10a}$$

$$A_{-,-}(n,m;x)P^{(\lambda)}_{n,m}(x) = \sqrt{(2\lambda+n+m-1)(2\lambda+n+m)}\,P^{(\lambda)}_{n-1,m-1}(x). \tag{10b}$$

The operators $A_{+,+}(n,m;x)$ and $A_{-,-}(n,m;x)$ simultaneously increase and decrease both of the indices, respectively.

3. The dynamical symmetry group $U(1,1)$ and infinite-fold degeneracy for Landau levels

Now by using a new variable r in the interval $0 < r < +\infty$, given by the following relation:

$$x = \frac{1}{\cosh\frac{r}{2}}, \tag{11}$$

and also the free variable ϕ in the interval $0 \leqslant \phi < 2\pi$, together with defining new parameter d as $d := n - m + 1$ with $d = 1, 2, 3, \ldots$, we can construct the irreducible representations for the Lie algebra $u(1, 1)$ so that its generators and the Casimir operator describe the dynamical symmetry group $U(1, 1)$ for moving a charged particle on the hyperbolic plane in the presence of a uniform magnetic field. In order to realize this goal, for given parameters λ and d we define the infinite-dimensional Hilbert space $\mathcal{H}^{(\lambda,d)} := \text{span}\{|{}^{\lambda}_{d,m}\rangle\}_{m\geqslant 0}$ which is generated by the following bases:

$$\left|{}^{\lambda}_{d,m}\right\rangle := \frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}} \frac{P^{(\lambda)}_{m+d-1,m}\left(\frac{1}{\cosh\frac{r}{2}}\right)}{h_{m+d-1}(\lambda)} \qquad \lambda > -1, \quad m = 0, 1, 2, \ldots \quad \text{and} \quad d = 1, 2, 3, \ldots. \tag{12}$$

Using equation (4), it becomes obvious that not only the bases of the Hilbert space $\mathcal{H}^{(\lambda,d)}$ with respect to the following inner product constitute an orthonormal set for different ms but also the bases of the Hilbert spaces $\mathcal{H}^{(\lambda,d)}$ with the different ds are orthogonal to each other with respect to the same inner product:

$$\begin{aligned}\left\langle{}^{\lambda}_{d,m}\middle|{}^{\lambda}_{d',m'}\right\rangle &= \int_{\phi=0}^{2\pi}\int_{r=0}^{\infty}\left(\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}\frac{P^{(\lambda)}_{m+d-1,m}\left(\frac{1}{\cosh\frac{r}{2}}\right)}{h_{m+d-1}(\lambda)}\right)^{*} \\ &\quad\times\left(\frac{\mathrm{e}^{\mathrm{i}m'\phi}}{\sqrt{2\pi}}\frac{P^{(\lambda)}_{m'+d'-1,m'}\left(\frac{1}{\cosh\frac{r}{2}}\right)}{h_{m'+d'-1}(\lambda)}\right)\frac{\sinh^{2\lambda+1}\frac{r}{2}}{\cosh^{2\lambda+2}\frac{r}{2}}\,\mathrm{d}r\,\mathrm{d}\phi \\ &= \delta_{dd'}\delta_{mm'}.\end{aligned} \tag{13}$$

The following irreducible representation for the Lie algebra $u(1, 1)$ in the Hilbert space $\mathcal{H}^{(\lambda,d)}$ can be immediately found using the representations (10*a*) and (10*b*) of the simultaneous laddering relations with respect to the different parameters n and m:

$$L_{+}^{(\lambda,d)}\left|{}^{\lambda}_{d,m-1}\right\rangle = \frac{h_{m+d-1}(\lambda)}{2h_{m+d-2}(\lambda)}\sqrt{(2\lambda+2m+d-2)(2\lambda+2m+d-1)}\left|{}^{\lambda}_{d,m}\right\rangle \tag{14a}$$

$$L_{-}^{(\lambda,d)}\left|{}^{\lambda}_{d,m}\right\rangle = \frac{h_{m+d-2}(\lambda)}{2h_{m+d-1}(\lambda)}\sqrt{(2\lambda+2m+d-2)(2\lambda+2m+d-1)}\left|{}^{\lambda}_{d,m-1}\right\rangle \tag{14b}$$

$$L_3\left|{}^{\lambda}_{d,m}\right\rangle = m\left|{}^{\lambda}_{d,m}\right\rangle \tag{14c}$$

$$I\left|{}^{\lambda}_{d,m}\right\rangle = \left|{}^{\lambda}_{d,m}\right\rangle, \tag{14d}$$

in which the explicit forms of the raising and the lowering operators corresponding to the parameter m, the Cartan subalgebra generator and the centre of algebra are, respectively,

$$L_{+}^{(\lambda,d)} = \mathrm{e}^{\mathrm{i}\phi}\left(\frac{\partial}{\partial r} + \mathrm{i}\coth r\frac{\partial}{\partial\phi} - \left(\lambda + \frac{d}{2}\right)\tanh\frac{r}{2}\right) \tag{15a}$$

$$L_{-}^{(\lambda,d)} = \mathrm{e}^{-\mathrm{i}\phi}\left(-\frac{\partial}{\partial r} + \mathrm{i}\coth r\frac{\partial}{\partial\phi} - \left(\lambda + \frac{d}{2} - \frac{1}{2}\right)\tanh\frac{r}{2} - \frac{2\lambda}{\sinh r}\right) \tag{15b}$$

$$L_3 = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{15c}$$

$$I = 1. \tag{15d}$$

Using the explicit form of the operators, one can easily conclude that the operators $\left\{L_+^{(\lambda,d)}, L_+^{(\lambda,d)}, L_3, I\right\}$ satisfy the commutation relations of the Lie algebra $u(1,1)$ as follows:

$$\left[L_+^{(\lambda,d)}, L_-^{(\lambda,d)}\right] = -2L_3 - \left(2\lambda + d - \tfrac{1}{2}\right)I \tag{16a}$$

$$\left[L_3, L_\pm^{(\lambda,d)}\right] = \pm L_\pm^{(\lambda,d)} \tag{16b}$$

$$\left[L_+^{(\lambda,d)}, I\right] = \left[L_-^{(\lambda,d)}, I\right] = [L_3, I] = 0. \tag{16c}$$

These commutation relations can also be verified by considering the representation of the Lie algebra $u(1,1)$ in the Hilbert space $\mathcal{H}^{(\lambda,d)}$ as relations (14a)–(14d). This Lie algebra $u(1,1)$ can decompose into the direct sum $su(1,1) \oplus u(1)$ if λ is chosen as $\lambda = -\frac{d}{2} + \frac{1}{4}$.

The Casimir operator of the Lie algebra $u(1,1)$ is given by

$$\begin{aligned} H^{(\lambda,d)} &= \frac{1}{2}\left[L_+^{(\lambda,d)} L_-^{(\lambda,d)} - L_3^2 - \left(2\lambda + d - \frac{3}{2}\right)L_3 + \frac{1}{2}\left(\lambda + \frac{d}{2} - 1\right)\right] \\ &= \frac{1}{2}\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{\sinh^2 r}\frac{\partial^2}{\partial\phi^2} - \left(\frac{2\lambda+1}{\sinh r} + \frac{\tanh\frac{r}{2}}{2}\right)\frac{\partial}{\partial r} + \mathrm{i}\left(\frac{-2\lambda}{\sinh^2 r} + \frac{2d-1}{4\cosh^2\frac{r}{2}}\right)\frac{\partial}{\partial\phi}\right. \\ &\quad \left. + \frac{1}{4}(2\lambda + d)\left(2\lambda - 1 + (d-1)\tanh^2\frac{r}{2}\right)\right]. \end{aligned} \tag{17}$$

It is possible to interpret the Casimir operator as a Hamiltonian operator corresponding to a charged particle on the hyperbolic plane described by the metric g_{ij}, in the presence of magnetic and electric fields with gauge connection A_i and electric potential V. In order to provide the mentioned interpretation it is sufficient to equalize the Casimir operator $H^{(\lambda,d)}$ with the general form of Laplace–Beltrami operator $\mathcal{L} = -\frac{1}{2}D_j^A D^{Aj} + V$, in which covariant derivative D_j^A is expressed in terms of gauge and Levi-Civita connections as $D_j^A = \nabla_j - \mathrm{i}A_j$ [17]. Here, the indices i and j take the values r and ϕ. Comparing the coefficients of the second order partial derivatives of the Casimir and Laplacian operators, the metric tensor corresponding to the hyperbolic plane can be calculated as

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 r \end{pmatrix}. \tag{18}$$

The non-vanishing components of Christoffel symbols and Ricci tensor of metric (18), as well as the Ricci scalar curvature, are derived as

$$\begin{aligned} &\Gamma^r_{\phi\phi} = -\tfrac{1}{2}\sinh 2r, \qquad \Gamma^\phi_{r\phi} = \coth r, \qquad R_{rr} = -1, \\ &R_{\phi\phi} = -\sinh^2 r, \qquad R = g^{ij}R_{ij} = -2, \end{aligned} \tag{19}$$

which describe a two-dimensional hyperbolic plane. The gauge potential A_i and the electric potential V, and consequently, their corresponding the constant magnetic and zero electric fields can be determined by comparing the coefficients of the first-order partial derivatives and the terms without derivative as

$$\left.\begin{aligned} A_r &= \tfrac{1}{4\mathrm{i}}\tfrac{\cosh r - 4\lambda - 1}{\sinh r} \\ A_\phi &= -\lambda + \tfrac{1}{2}\left(d - \tfrac{1}{2}\right)(\cosh r - 1) \\ V &= \tfrac{1}{2}\left(\lambda^2 + \lambda d - \lambda - \tfrac{d}{4} - \tfrac{1}{4}\right) \end{aligned}\right\} \Longrightarrow \begin{cases} \mathbf{B} = \frac{\partial A_\phi}{\partial r}\,\mathrm{d}r \wedge \mathrm{d}\phi = \frac{1}{2}\left(d - \frac{1}{2}\right)\sinh r\,\mathrm{d}r \wedge \mathrm{d}\phi \\ \mathbf{E} = -\frac{\partial V}{\partial r}\,\mathbf{e}_r = 0. \end{cases} \tag{20}$$

Since $\sinh r\, \mathrm{d}r \wedge \mathrm{d}\phi$ is the volume form of hyperbolic plane, we thus deal with a uniform magnetic field on it. Now, it is easy to show that the Casimir operator $H^{(\lambda,d)}$ satisfies the following eigenvalue equation on the Hilbert space $\mathcal{H}^{(\lambda,d)}$:

$$H^{(\lambda,d)} \left|{}^{\lambda}_{d,m}\right\rangle = \frac{1}{2}\left(\lambda + \frac{d}{2}\right)\left(\lambda + \frac{d}{2} - 1\right)\left|{}^{\lambda}_{d,m}\right\rangle \qquad m = 0, 1, 2, \ldots. \tag{21}$$

Therefore for given λ and d, the kets $\left|{}^{\lambda}_{d,m}\right\rangle$ as Landau levels describe the motion of a spinless charged particle on the hyperbolic plane in the presence of a uniform magnetic field with the dynamical symmetry group $U(1, 1)$ and the infinite-fold degeneracy in the Hilbert representation space $\mathcal{H}^{(\lambda,d)}$ via the discrete quantum number m. While the strength of magnetic field is independent of the parameter λ, it is quantized in terms of d as odd multiples $2d - 1$. This condition is equivalent to a generalized quantization condition for the uniform magnetic field. Relations (20) show that if we cancel the term $\frac{1}{2}(\lambda^2 + \lambda d - \lambda)$ of both sides of the eigenvalue equation (21), then we obtain an additional degeneracy for spectrum via the continuous parameter λ on all Hilbert spaces $\mathcal{H}^{(\lambda,d)}$ with a given strength of the magnetic field [5, 6]. Furthermore, for a fixed λ, the Hilbert spaces $\mathcal{H}^{(\lambda,d)}$ with the different ds are allocated to the uniform magnetic fields with different strengths.

4. $u(2)$ maps between the Landau levels Hilbert spaces with different strengths of the magnetic field

Now we are going to realize the representation of the compact Lie algebra $u(2)$ via simultaneous shift of both indices m and d. This realization is done by shifting between the orthogonal Hilbert spaces $\mathcal{H}^{(\lambda,d)}$ with the different ds. Using the raising and lowering relations (6a) and (6b), it can be easily shown that the quantum states $\left|{}^{\lambda}_{d,m}\right\rangle$ also satisfy the following relations:

$$J_{-+}^{(\lambda)} \left|{}^{\lambda}_{d+1,m-1}\right\rangle = \sqrt{d(2\lambda + 2m + d - 1)}\left|{}^{\lambda}_{d,m}\right\rangle \tag{22a}$$

$$J_{+-}^{(\lambda)} \left|{}^{\lambda}_{d,m}\right\rangle = \sqrt{d(2\lambda + 2m + d - 1)}\left|{}^{\lambda}_{d+1,m-1}\right\rangle \tag{22b}$$

$$J_3 \left|{}^{\lambda}_{d,m}\right\rangle = m\left|{}^{\lambda}_{d,m}\right\rangle \tag{22c}$$

$$I \left|{}^{\lambda}_{d,m}\right\rangle = \left|{}^{\lambda}_{d,m}\right\rangle, \tag{22d}$$

where the explicit forms of the generators are, respectively,

$$J_{-+}^{(\lambda)} = \mathrm{e}^{\mathrm{i}\phi}\left(-2\cosh\frac{r}{2}\frac{\partial}{\partial r} - \frac{\mathrm{i}}{\sinh\frac{r}{2}}\frac{\partial}{\partial\phi}\right) \tag{23a}$$

$$J_{+-}^{(\lambda)} = \mathrm{e}^{-\mathrm{i}\phi}\left(2\cosh\frac{r}{2}\frac{\partial}{\partial r} - \frac{\mathrm{i}}{\sinh\frac{r}{2}}\frac{\partial}{\partial\phi} + \frac{2\lambda}{\sinh\frac{r}{2}}\right) \tag{23b}$$

$$J_3 = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{23c}$$

$$I = 1. \tag{23d}$$

One can easily conclude that the operators $\left\{J_{-+}^{(\lambda)}, J_{+-}^{(\lambda)}, J_3, I\right\}$ satisfy the commutation relations of Lie algebras $u(2)$ as follows:

$$\left[J_{-+}^{(\lambda)}, J_{+-}^{(\lambda)}\right] = 2J_3 + 2\lambda I \tag{24a}$$

$$\left[J_3, J_{\mp\pm}^{(\lambda)}\right] = \pm J_{\mp\pm}^{(\lambda)} \tag{24b}$$

$$\left[J_{-+}^{(\lambda)}, I\right] = \left[J_{+-}^{(\lambda)}, I\right] = [J_3, I] = 0. \tag{24c}$$

In the special case that $\lambda = 0$, the associated Gegenbauer functions $P^{(\lambda)}_{m+d-1,m}\left(\frac{1}{\cosh\frac{r}{2}}\right)$ are converted to the Legendre functions and the Lie algebra $u(2)$ decomposes into the direct sum $su(2) \oplus u(1)$. It is also clear that the representation (22) for Lie algebra $u(2)$ is finite, since the indices d and m cannot simultaneously take values smaller than 1 and 0, respectively. In fact, for every given $d + m$, relations (22) present a $(d + m)$-dimensional representation of the Lie algebra $u(2)$. According to equations (22*a*) and (22*b*), the operator $J^{(\lambda)}_{-+}$ maps the Hilbert space $\mathcal{H}^{(\lambda,d+1)}$ into $\mathcal{H}^{(\lambda,d)}$, while the operator $J^{(\lambda)}_{+-}$ does the opposite. This means that it is possible to go from Landau levels of a given field strength to the other one just using the generators of Lie algebra $u(2)$.

Equation (22*a*) shows that the lowest states $\left|{}^{\lambda}_{1,m}\right\rangle$ of $u(2)$ belonging to the Hilbert space $\mathcal{H}^{(\lambda,1)}$ satisfies a first-order differential equation as $J^{(\lambda)}_{-+}\left|{}^{\lambda}_{1,m}\right\rangle = 0$ with the following solution:

$$\left|{}^{\lambda}_{1,m}\right\rangle = \frac{(-1)^m\sqrt{\Gamma(2\lambda+2m+2)}}{2^{\lambda+m+\frac{1}{2}}\Gamma(\lambda+m+1)}\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}\tanh^m\frac{r}{2} \qquad m = 0, 1, 2, \ldots. \tag{25}$$

So, with the help of (22*b*), one can obtain an arbitrary Landau level of the space $\mathcal{H}^{(\lambda,d)}$ in the framework of the algebraic manner as

$$\left|{}^{\lambda}_{d,m}\right\rangle = \sqrt{\frac{\Gamma(2\lambda+2m+d)}{\Gamma(d)\Gamma(2\lambda+2m+2d-1)}}\left(J^{(\lambda)}_{+-}\right)^{d-1}\left|{}^{\lambda}_{1,m+d-1}\right\rangle \qquad d = 1, 2, 3, \ldots, \quad m = 0, 1, 2, \ldots. \tag{26}$$

Equation (26) implies that all bases of Hilbert spaces $\mathcal{H}^{(\lambda,d)}$ are generated by successive action of the operator $J^{(\lambda)}_{+-}$ on the lowest states $\left|{}^{\lambda}_{1,m}\right\rangle$. Therefore, we have obtained an additional symmetry, the so-called $u(2)$ on the Landau levels, so that it is realized by the quantum states with different strengths.

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